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1999 J. Phys. A: Math. Gen. 32 L307

(http://iopscience.iop.org/0305-4470/32/27/103)

J. Phys. A: Math. Gen. **32** (1999) L307–L314. Printed in the UK PII: S0305-4470(99)03875-5

LETTER TO THE EDITOR

The associated Camassa–Holm equation and the KdV equation

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Received 30 April 1999, in final form 24 May 1999

Abstract. Recently the associated Camassa–Holm (ACH) equation, related to the Fuchssteiner–Fokas–Camassa–Holm equation by a reciprocal transformation, was introduced by Schiff, who derived Bäcklund transformations by a loop group technique and used these to obtain some simple soliton and rational solutions. We show how the ACH equation is related to Schrödinger operators and KdV, and describe how to construct solutions of ACH from tau-functions of the KdV hierarchy. Rational, N-soliton and elliptic solutions are considered, as well as exact solutions given by a particular case of the third Painlevé transcendent.

1. Introduction

A great deal of interest has been generated by the Fuchssteiner–Fokas–Camassa–Holm (FFCH) equation,

$$u_T = 2f_X u + f u_X \qquad u = \tfrac{1}{2} f_{XX} - 2f \tag{1.1}$$

(for comparison we take the choice of coefficients in [27]), which originally appeared in the work of Fuchssteiner and Fokas [12], but was later derived as an equation for shallow water waves by Camassa and Holm [6]; recently it has been shown to be a particular case of a class of models for ideal fluids [17]. Of particular importance was the discovery [6] that (1.1) admits peaked solitons or 'peakons', described in terms of solutions of an associated integrable finite-dimensional dynamical system which has subsequently been related to the Toda lattice [5]. Although the FFCH equation is integrable, and has been shown [13] to be related by a reciprocal transformation to the first negative flow of the KdV hierarchy (also known as the AKNS equation [1]),

$$R(U)U_t = 0 \qquad R(U) = \partial_x^2 + 4U + 2U_x\partial_x^{-1} \tag{1.2}$$

it has many non-standard features (for instance, it only possesses the weak Painlevé property [15]) and there is still much to be understood about its solutions.

Inspired by [13], Schiff introduced the associated Camassa–Holm (ACH) equation [27]

$$p_t = p^2 f_x \qquad f = \frac{p}{4}(\log p)_{xt} - \frac{p^2}{2} \tag{1.3}$$

which (for positive u) has a one to one correspondence with solutions of the FFCH equation (1.1) given by

$$p = \sqrt{u} \qquad \mathrm{d}x = p\,\mathrm{d}X + pf\,\mathrm{d}T \qquad \mathrm{d}t = \mathrm{d}T \tag{1.4}$$

0305-4470/99/270307+08$30.00

(the independent variables x, t of ACH are denoted t_0, t_1 in [27]); a solution of ACH where p has zeros corresponds to a number of solutions of (1.1) where u has fixed sign. For more details on reciprocal transformations such as (1.4) see [11, 25]. In [27], a loop group interpretation was given for (1.3), making use of the fact that it is the (zero curvature) compatibility condition for the linear system

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = \begin{pmatrix}0 & 1/p\\ p/\lambda + 1/p & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{1.5}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = \begin{pmatrix}-p_t/2p & \lambda\\ \lambda - 2f & p_t/2p\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{1.6}$$

and it was indicated that this ACH equation is part of an integrable hierarchy of zero-curvature equations. Automorphisms of the loop group were used to derive two Bäcklund transformations (BTs) for (1.3), these were applied to the constant background solution $p = h$ to obtain some simple solutions, and by applying the hodograph transformation (1.4) apparently novel solutions of the FFCH equation (1.1) were generated.

In the following we show that since the hodograph transformation is essentially the same as in [13], the ACH equation is naturally related to the inverse KdV equation (1.2) and has a Lax pair of which one part is just a (time-independent) Schrödinger equation. This connection yields a straightforward derivation of the BTs presented in [27], and further leads to a simple recipe for constructing solution of ACH from tau-functions of the KdV hierarchy. Some of these results first appeared in [19].

2. Basic properties of ACH equation

2.1. Lax pair and tau-function

The ACH equation (1.3) may be rewritten as

$$U_t = -2p_x \qquad U = -\frac{1}{2}\left(\frac{pp_{xx} - \frac{1}{2}p_x^2 + 2}{p^2}\right). \tag{2.1}$$

This leads directly to a connection with the KdV hierarchy. In fact, by writing $p = -\frac{1}{2}\partial_x^{-1}U_t$, it follows that U satisfies the inverse KdV equation (1.2). This suggests that there should be a direct link with Schrödinger operators, and indeed this is the case†.

Setting $\phi = p^{\frac{1}{2}}\psi_1$ (and using $\psi_2 = p\psi_{1,x}$) in (1.5), (1.6) leads to the Lax pair

$$(\partial_x^2 + U - 1/\lambda)\phi = 0 \tag{2.2}$$

$$\phi_t = \lambda(p\phi_x - \tfrac{1}{2}p_x\phi). \tag{2.3}$$

The two compatibility conditions for this Lax pair (assuming that U is as yet undefined) are simply the first equation for U_t in (2.1) and

$$(\partial_x^3 + 4U\partial_x + 2U_x)p = 0 \tag{2.4}$$

and it is an immediate consequence that U satisfies (1.2). The latter third-order equation for p can be integrated once to give

$$pp_{xx} - \tfrac{1}{2}p_x^2 + 2Up^2 + F(t) = 0. \tag{2.5}$$

Equation (2.5) is known as the Ermakov–Pinney equation (see [9] and references). The function $F(t)$ is arbitrary, but it is clear that the ACH equation (with the definition of U as in (2.1))

† I am grateful to Decio Levi and Orlando Ragnisco for pointing this out.

corresponds to the particular choice $F = 2$. Provided $F \neq 0$ it is always possible to rescale p by $\sqrt{F(t)/2}$ and redefine t to obtain the form (2.1).

We can define the tau-function σ of ACH by

$$p(x,t) = -(\log\sigma)_{xt}.$$

From the ACH equation in the form (2.1) we can integrate once to find the standard KdV formula for the potential of the Schrödinger operator:

$$U(x,t) = 2(\log\sigma)_{xx}.$$

Substituting for p, U in (2.4) yields a trilinear equation in x, t for the tau-function σ, namely

$$\sigma_{4x,t}\sigma^2 - 4\sigma_{3x,t}\sigma_x\sigma + 2\sigma_{xxt}\sigma_{xx}\sigma - \sigma_t\sigma_{4x}\sigma + 4\sigma_{xxt}\sigma_x^2 - 4\sigma_{xt}\sigma_x\sigma_{xx} + 4\sigma_t\sigma_x\sigma_{3x} \\ -2\sigma_t\sigma_{xx}^2 = 0$$

which is a reduction of the trilinear appearing in [28]. Hirota and Satsuma [16] obtained bilinear equations for the tau-function of the AKNS equation, but this required the introduction of another independent variable (corresponding to the t_3 flow of KdV). The result proved in section 3 below, implies that such σ may be found by applying a gauge transformation to a tau-function of the KdV hierarchy, $\tau = \exp[x^2/(4h^2) + hxt]\sigma$.

2.2. Bäcklund transformations

By the use of loop group techniques, Schiff [27] derived two BTs for the ACH equation. The first of these BTs is

$$\tilde{p} = p - s_x \qquad s_x = -(p\lambda)^{-1}s^2 + \lambda p^{-1} + p \qquad s_t = -s^2 + (\log p)_t s + \lambda(\lambda - 2f) \tag{2.6}$$

where λ is a Bäcklund parameter. A superposition principle was also found for this BT, leading to a formula for the two-soliton solution of ACH. Schiff's second BT may be written (after some simplification) as

$$\tilde{p} = p - (\log\chi)_{xt} \qquad (pB_x)_x = B(p^{-1} + p\lambda^{-1}) \qquad B_t = -\tfrac{1}{2}(\log p)_t B + p\lambda B_x \tag{2.7}$$

where χ is determined from the first-order equations

$$\chi_x = p\lambda^{-1}B^2 \qquad \chi_t = \lambda(p^2B_x^2 - B^2).$$

We observe that (as also noted in [27]) the equations for B in (2.7) follow from the linearization of the first Riccati equation in (2.6) via the substitution $s = p\lambda(\log B)_x$; this linearization is just the linear problem (1.5) when we identify $B = \psi_1$. The second BT actually gives nothing new, as it is equivalent to applying (2.6) twice with the same Bäcklund parameter each time. So we concentrate on the first BT, noting that the second Riccati equation in (2.6) is linearized by the substitution $s = (\log\phi)_t$, and then we see that the BT (2.6) is equivalent to

$$\tilde{p} = p - (\log\phi)_{xt}$$

where ϕ is a solution of the Lax pair (2.2), (2.3). In fact, a tedious direct calculation shows that under this BT the potential of the Schrödinger operator becomes

$$\tilde{U} = U + 2(\log\phi)_{xx}$$

and also that $\tilde{\phi} = \phi^{-1}$ is a solution of the same Lax pair with U, p replaced by $\tilde{U}, \tilde{p}$. Hence the first BT (2.6) derived by Schiff is equivalent to the well known Crum transformation obtained by factorization of the Schrödinger operator [2], which gives the standard Darboux–Bäcklund transformation for the KdV hierarchy.

3. Solutions of ACH from KdV

The KdV hierarchy (see e.g. [22]) is the sequence of evolution equations

$$q_{t_{2j-1}} = 2(P_j[q])_{t_1} \tag{3.1}$$

(for $j = 1, 2, 3, \ldots$), which arise as the compatibility condition for the Schrödinger equation

$$(\partial_{t_1}^2 + q)\phi = \mu^2\phi \tag{3.2}$$

with the sequence of linear problems

$$\phi_{t_{2j+1}} = \Pi_j \phi_{t_1} - \tfrac{1}{2}\Pi_{j,t_1}\phi \qquad \Pi_j[q;\mu] := \sum_{k=0}^{j} P_{j-k}[q]\mu^{2k} \qquad P_0 = 1 \tag{3.3}$$

(for $j = 0, 1, 2, \ldots$). The t_{2j-1} are the times of the hierarchy (the odd times of the KP hierarchy [23]), and the differential polynomials $P_k[q]$ are the Gelfand–Dikii polynomials [14], which can be defined recursively using a form of the Ermakov–Pinney equation (2.5). Also, in terms of the tau-function $\tau(t_1, t_3, t_5, \ldots)$ of the hierarchy, q and the P_j are given by

$$q = 2(\log\tau)_{t_1 t_1} \qquad P_j = (\log\tau)_{t_1 t_{2j-1}}. \tag{3.4}$$

This tau-function satisfies a sequence of bilinear equations [23], but we shall not make use of these here.

To make the connection with the ACH equation we simply observe that for a solution of ACH satisfying $p \to h$ at infinity, it is clear that U as defined in (2.1) satisfies $U \to -1/h^2$. Thus, considering the Schrödinger equation (2.2) with such a potential U is instead equivalent to taking a Schrödinger equation (3.2) with potential q vanishing at infinity, when we identify

$$q = U + 1/h^2 \qquad \mu = \sqrt{\frac{1}{h^2} + \frac{1}{\lambda}}. \tag{3.5}$$

This suggests the following proposition.

Proposition. *Given a tau-function $\tau(t_1, t_3, t_5, \ldots)$ for a solution $q(t_1, t_3, t_5, \ldots)$ of the KdV hierarchy, a corresponding solution of the ACH equation (2.1) is given by*

$$p = h - (\log\tau)_{xt} \qquad t_1 = \tilde{t}_1 + x - h^3 t \qquad t_{2j+1} = \tilde{t}_{2j+1} - h^{2j+3}t \qquad (j \geqslant 1) \tag{3.6}$$

(with $\tilde{t}_{2j+1}$ independent of x, t). The corresponding Schrödinger potential is given by

$$U = q - 1/h^2 = -1/h^2 + 2(\log\tau)_{xx}.$$

This implies that vanishing solutions of KdV (with $q \to 0$ as $|t_1| \to \infty$) yield solutions of ACH on constant background h, but the result is valid for non-vanishing q as well. The proof of the proposition is very straightforward, for using (3.6) and (3.4) we see that we can write

$$p = h + \sum_{j=1}^{\infty} h^{2j+1}(\log\tau)_{t_1 t_{2j-1}} = \sum_{j=0}^{\infty} P_j h^{2j+1}.$$

Series of this type are a standard tool for obtaining recursive formulae for the flows of integrable hierarchies based on Schrödinger operators (see [3], for instance). The right-hand equation in (2.1) (the Ermakov–Pinney equation) is naturally rewritten as

$$pp_{t_1 t_1} - \tfrac{1}{2}p_{t_1}^2 + 2(q - 1/h^2)p^2 + 2 = 0$$

and by expanding in powers of h the recursion relations for the Gelfand–Dikii polynomials P_j are obtained. Hence p, as defined above, is automatically a solution of this Ermakov–Pinney equation, and writing everything in terms of the tau-function the ACH equation itself (the left-hand equation in (2.1)) is just the tautology $2(\log\tau)_{xxt} = 2(\log\tau)_{xtx}$.

It is also fairly simple to show that, provided μ is identified as in (3.5), ϕ satisfying (3.2) and the sequence of linear problems (3.3) provides a solution to the ACH Lax pair (2.2), (2.3) (thus providing an alternative proof of the proposition). In order to show that (2.3) is satisfied, it is necessary to write

$$\phi_t = -\sum_{j=0}^{\infty} h^{2j+3}\phi_{t_{2j+1}} = -\sum_{j=0}^{\infty} h^{2j+3}(\Pi_j\phi_{t_1} - \tfrac{1}{2}\Pi_{j,t_1}\phi)$$

expanding each Π_j in μ and resumming by use of the geometric series $\lambda = -h^2(1-\mu^2h^2)^{-1} = -\sum_{k=0}^{\infty} h^{2j+2}\mu^{2j}$; noting that t_1 derivatives may be replaced by x derivatives, (2.3) results.

4. Exact solutions

4.1. Rational and soliton solutions

Rational and soliton solutions are easily obtained with the use of the above proposition. Rational solutions of KdV correspond to tau-functions $\tau^{(k)}$ which are most easily expressed as Wronskian determinants of odd Schur polynomials,

$$\tau^{(k)} = [p_{2k-1}, p_{2k-3}, \ldots, p_1]$$

for $k = 1, 2, 3, \ldots$. The sequence of Schur polynomials may be defined by a generating function, $\sum_{l=0}^{\infty} p_l\nu^l = \exp(\sum_{j=1}^{\infty} t_j\nu^j)$ (see e.g. [23]). The above Wronskians are independent of the even times t_{2k}, and we are using the notation $[\ldots]$ to denote the Wronskian as in [18]. Thus, we find the sequence of rational solutions of ACH,

$$p^{(k)} = h - (\log\tau^{(k)})_{xt}$$

where the t_j are given in terms of x, t as in (3.6). After rescaling these $\tau^{(k)}$ are the same as the Adler–Moser polynomials θ_k [2] obtained by application of the Crum transformation. Hence these rational solutions of ACH may also be obtained by repeated use of Schiff's first BT (2.6) with Bäcklund parameter $\lambda = -h^2$, starting from the trivial solution $p = h$.

It is also well known that soliton tau-functions can be written as Wronskian determinants [23, 26]. For KdV the N-soliton tau-function is built out of N functions η_j of the form

$$\eta_j = \exp\xi(t_1, t_3, \ldots; \mu_j) + c_j\exp\xi(t_1, t_3, \ldots; -\mu_j)$$

where $\xi(t_1, t_3, \ldots; \mu) = \sum_{k=1}^{\infty} t_{2k-1}\mu^{2k-1}$. Using the expressions (3.6) for the t_{2j-1}, and defining $\lambda_j = -h^2(1-\mu_j^2h^2)^{-1}$ (which may be written as a geometric series as in the previous section), we find

$$\eta_j = \exp\left(\sqrt{\frac{1}{h^2}+\frac{1}{\lambda_j}}(x - h\lambda_j t + x_j)\right) + c_j\exp\left(-\sqrt{\frac{1}{h^2}+\frac{1}{\lambda_j}}(x - h\lambda_j t + x_j)\right)$$

for x_j, c_j constants. Then the N-soliton solution of the ACH equation may be written as

$$p(N) = h - (\log W(N))_{xt} \qquad W(N) = [\eta_1, \eta_2, \ldots, \eta_N]$$

(where all the t_1 derivatives from the KdV Wronskian formula [26] may be replaced by x derivatives). The N-soliton solutions of the AKNS equation found in [16] are obtained from the above formulae in the particular case $h = 1$.

4.2. Elliptic solutions

The general travelling wave solution of the ACH equation (also mentioned in [27]) is given in terms of the Weierstrass $\wp$-function [29],

$$p(x,t) = -c(\wp(x-ct) - \wp(\kappa)) \tag{4.1}$$

with κ constant such that $\wp'(\kappa) = 2c^{-1}$. For this solution we have $U = -2\wp(x-ct) - \wp(\kappa)$. The corresponding travelling wave solution of the the FFCH equation (1.1) is

$$f(X,T) = -(\wp(\rho) - \wp(\kappa))^{-1} - \frac{c^2}{4}\wp''(\kappa)$$

where (by standard identities for Weierstrass functions [29]) $\rho = \rho(X,T)$ is determined from

$$X - \frac{c^2}{4}\wp''(\kappa)T = -\zeta(\kappa)\rho + \frac{1}{2}\log\frac{\boldsymbol{\sigma}(\kappa+\rho)}{\boldsymbol{\sigma}(\kappa-\rho)}.$$

In the light of known results on KdV [21], the form of the solution (4.1) suggests that we should consider the ansatz

$$p = -\sum_{j=1}^{N}\dot{x}_j\wp(x-x_j) + k \tag{4.2}$$

where the poles $x_j = x_j(t)$ depend on time, k is constant and $\dot{x}_j = \frac{\mathrm{d}}{\mathrm{d}t}x_j$. Then we take

$$U = -2\sum_{j=1}^{N}\wp(x-x_j) + \ell$$

(for constant ℓ) and substitute in (2.4). After some cancellation the resulting equation has third-order, double and simple poles at $x = x_j$ for each j. The residues at the third-order poles yield the matrix equation

$$Mv = ke \tag{4.3}$$

where M is the symmetric matrix with entries

$$M_{jj} = -\ell + 2\sum_{k\neq j}\wp(x_j - x_k) \qquad M_{jk} = \wp(x_j - x_k) \qquad (j \neq k)$$

and $v = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N)^T$, $e = (1, 1, \ldots, 1)^T$. At the second-order poles we find

$$\sum_{k\neq j}\wp'(x_j - x_k) = 0 \tag{4.4}$$

for each j, while the residue at the simple pole is just the t derivative of this constraint.

Comparing with [21] we see that (4.4) are just the standard KdV constraints on the elliptic Calogero–Moser system, corresponding to stationarity with respect to the t_2 flow generated by the Hamiltonian

$$H_2 = \tfrac{1}{2}\sum_{j=1}^{N}\pi_j^2 + g^2\sum_{j,k}{}'\wp(x_j - x_k)$$

(π_j denote momenta, g^2 is a coupling constant, and $\sum'$ means sum with $j \neq k$). So we see that ACH admits elliptic solutions of the form (4.2) with poles moving according to a constrained Calogero–Moser system (4.3), which must correspond to the first negative Calogero–Moser flow subject to the KdV constraints (4.4). A detailed study of the solutions of these constraints has been made recently by Deconinck and Segur [10].

4.3. Solutions in terms of PIII

The ACH equation has a scaling similarity reduction

$$p = (2t)^{-\frac{1}{2}} w(z) \qquad z = (2t)^{\frac{1}{2}} x$$

where $w(z)$ satisfies the ODE

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{1}{z}(2w^2 + \beta) - \frac{4}{w} \tag{4.5}$$

$'$ denotes $\mathrm{d}/\mathrm{d}z$ and β is an arbitrary constant. This ODE is a special case of the Painlevé transcendent PIII, which (with the standard form taken in [4]) corresponds to the choice of parameters $\alpha = 2, \gamma = 0, \delta = -4$ (and β remaining arbitrary). The symmetry reduction of the AKNS equation to PIII has also been found in [8]. The corresponding solution of the FFCH equation (1.1) is given by

$$f(X, T) = (2T)^{-1}(\beta/4 - z/w(z))$$

with $z = z(X, T)$ determined implicitly from

$$X + \frac{\beta}{8} \log T = \int^z \frac{\mathrm{d}y}{w(y)}.$$

PIII has a large number of BTs, and some special exact solutions, which are systematically catalogued in [4]. However, for the choice of parameters relevant here, the only BTs that survive are one referred to in [4] as transformation V,

$$\tilde{w} = \frac{zw'}{w^2} - \frac{\beta + 2}{2w} + \frac{2z}{w^2}$$

together with its inverse

$$\bar{w} = -\frac{zw'}{w^2} - \frac{\beta - 2}{2w} + \frac{2z}{w^2}.$$

These BTs send a solution w to (4.5) for parameter β to another solution $\tilde{w}$, $\bar{w}$ with parameter $\beta + 4$, $\beta - 4$, respectively, and may be obtained by reduction of the BT (2.6) (the Crum transformation) in the case $\lambda = \infty$.

For this choice of parameter values there is a hierarchy of special solutions rational in $z^{\frac{1}{3}}$, which can be obtained by applying the BTs to the seed solution $w_0 = (2z)^{\frac{1}{3}}$ for $\beta = 0$. For example, for $\beta = \pm 4$ (4.5) admits the particular solutions

$$w_{\pm 4} = 2^{\frac{1}{3}} z^{\frac{1}{3}} \mp \tfrac{1}{3} 2^{\frac{2}{3}} z^{-\frac{1}{3}}.$$

Other solutions in this hierarchy can be obtained from table 6 of [4] (on setting the parameters $\mu = 2, \kappa = 2^{\frac{1}{3}}$).

Okamoto [24] has introduced Hamiltonians for the Painlevé equations, given as the logarithmic derivative of a tau-function. However, the case $\gamma = 0$ of PIII relevant here is excluded in [24]. Nevertheless, it is still possible to take the Hamiltonian

$$H = z^{-1} w^2 \pi^2 + (2 - \tfrac{1}{2}(\beta - 2) w z^{-1})\pi - w$$

for this degenerate case (with π denoting the momentum conjugate to w), with tau-function $\tau(z)$ such that

$$H(z) = \tfrac{\mathrm{d}}{\mathrm{d}z} \log \tau(z).$$

This is not quite the same as the tau-function σ introduced in section 2 (we conjectured this in [19]). For the similarity reduction we require $\sigma = \sigma(z)$ and then we find

$$w = -z(\log \sigma)'' - (\log \sigma)' \qquad \bar{w} = -2\pi = -z(\log \bar{\sigma})'' - (\log \bar{\sigma})'$$

(where $\bar{w}$ is obtained by the action of the second BT above), which leads to

$$\tau = \sigma \bar{\sigma}.$$

5. Conclusions

We have shown how the ACH equation introduced by Schiff is related to the KdV hierarchy, and used this connection to construct a variety of exact solutions. We have also found solutions in terms of a particular case of the Painlevé transcendent PIII. By the use of the hodograph transformation (1.4) these solutions of the ACH equation yield solutions of the FFCH equation (1.1), and it would be interesting to study the transformed solutions. We have also found [20] that similar methods apply to the $2+1$ generalization of the FFCH equation introduced in [7].

It is a pleasure to thank Orlando Ragnisco, Decio Levi, Andrew Pickering and Jeremy Schiff for useful discussions. I would also like to thank Sergei Manakov and others present at the Integrable Systems Seminar of Roma 'La Sapienza' for helpful comments. I am very grateful to the Leverhulme Trust for giving me a Study Abroad Studentship in Rome, and to the ARC of Australia for my International Research Fellowship in Adelaide.

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